Quantum Tests of the Foundations of General Relativity*

Claus Lämmerzahl†

Laboratoire de Gravitation et Cosmologie Relativistes,

Université Pierre et Marie Curie, CNRS/URA 769, 75252 Paris Cedex 05, France

and

Fakultät für Physik der Universität Konstanz, D - 78434 Konstanz, Germany

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Abstract

A new test theory for describing tests of fundamental principles of Special and General Relativity is presented. Using a generalised Pauli equation which may be based on a generalised Dirac equation, possible violations of local Lorentz invariance and local position invariance on the quantum level are described. On the quantum level there are more possibilities to violate these principles than on the classical level. The corresponding terms can be tested by Hughes–Drever type experiments or by atom interferometry. It is proposed that an atom interferometric test of Local Position Invariance will give a three order improvement of existing estimates.

1 Introduction

General Relativity (GR) is based on Einstein's Equivalence Principle (EEP) which implies that gravitation is a metric theory. EEP consists of the weak equivalence principle (WEP), local Lorentz invariance (LLI) and local position invariance (LPI); for a review see [1, 2]. All these notions essentially rely on point particles and paths. Therefore GR is primarily applicable to the motion of satellites, planets and light rays. However, since quantum matter possesses more degrees of freedom and is always spread out at least over a certain space-like region, a theory of gravity at the quantum level may be different from GR at the classical level and therefore should be founded on principles using quantum mechanics only. It may be possible, for example, that gravity on the quantum level has to be described by more fields than the metrical field. In addition, for doing tests of the foundations of GR on the quantum level, one should take into account that there may be more possibilities to break LLI and LPI on the quantum level than on the classical level.

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†e-mail: Claus@spock.physik.uni-konstanz.de

Therefore, in order to test gravitational theories at the quantum level it is necessary to use a test theory which is based solely on quantum notions. We do not transcribe classical notions into the quantum domain, e.g. introduce a PPN metric or the corresponding γ–matrices into the Klein–Gordon or Dirac equation. One should allow the most general interaction on the quantum level. We base our test theory on a generalised Pauli equation (GPE) which on the one hand is general enough to allow violations of WEP, LLI, and LPI, and on the other hand still obeys fundamental quantum principles. An important point is that we include spin which is necessary since there are no massive particles without spin.

An important point of our considerations is that in order to arrive at our test theory we do not use any geometrical notion. This is necessary from a logical point of view since we want to reach the result, that only under certain circumstances (experimental results), gravitation can be described by means of a space–time geometry. In addition, we do not use a Lagrangian formalism.

Our test theory is a test theory for both special relativity *and* GR together. In contrast to the test theory of Robertson [3] and Mansouri–Sexl [4], and also their gravitationally modified form [5], our theory does not rely on a selected frame of reference. It is the *dynamics* of the quantum field which is responsible for whether LLI is violated or not. The coupling to gravity is accomplished by introducing the Newtonian potential in a most general way which may be particle dependent.

There are other test theories, like the $TH\epsilon\mu$ [6, 1] formalism, which allows a difference in the velocity of light and the maximum speed of massive particles. The main difference between this and our test theory, is that in our case the violation of the EEP is due to the structure of the dynamical equation of one matter field, and not due to an 'anomalous' interaction between matter and the electromagnetic field. In this sense our test theory is not an alternative to the $TH\epsilon\mu$–formalism, but instead a theory of one single multicomponent matter field. The structure of the coupling of the GPE (or of the generalised Dirac equation on which we base our GPE) to the electromagnetic field is another question. It may indeed happen that both fields, the GPE and the Maxwell field, are dynamically incompatible in the sense that they possess to two different maximal velocities of propagation which of course leads to certain observable consequences. It is the $TH\epsilon\mu$ formalism which describes the experimental consequences due to this difference in the maximum speed in, for example, atomic systems. Here we skip this point and assume the usual minimal coupling to the electromagnetic field obeying the usual Maxwell equations.

Best estimates for parameters characterising the violation of LLI and LPI are given by Hughes–Drever type experiments and by the Vessot–Levine rocket red shift experiment (for a recent review see [2]). LPI and LLI violating parameters are constraint by 10^{-4} and 10^{-30}, respectively. Here we want to indicate that in some cases more precise tests are possibleusing atom beam interferometers of the Kasevich–Chu type [20, 21]. If one carries through atom beam interferometric tests in the field of the earth with two different types of atoms, then the accuracy of these devices may test the validity of the WEP in the quantum domain with an accuracy 10^{-8} (it is announced that these interferometers may be improved in the near future) and may especially restrict the LPI violating parameter to an accuracy of about 10^{-7} which will be an imoprovement of the present estimate by three orders. Hughes–Drever type experimenbts are already carried through, and we use a reanalysis of these experiments for

calculating constraints for the parameters of our test theory. In addition, since all experiments made on earth are in fact done in a noninertial frame moving in a gravitational field, each high precision experiment can be used as a test of the underlying space–time structure.

In the following we treat the GPE which we derive from a non–relativistic limit of a generalised Dirac equation (GDE). Then we calculate some experiments, namely atom beam interferometry [7], and the modification of energy levels of atoms, and give estimates for the various parameters describing LLI– and LPI–violation.

2 The generalised Pauli equation

2.1 The generalised Dirac equation

We base our non–relativistic test theory on a generalised Dirac equation (see, for example, [8, 9])

$$i\partial_t\varphi = -ic(\widetilde{\alpha}^i\nabla_i + i\Gamma)\varphi + mc^2\widetilde{\beta}\varphi + e\phi\varphi \tag{1}$$

$(i, j = 1, 2, 3)$ where φ is a complex 4–component field (it is possible to carry through the following consideration for higher component fields too; however, we want to restrict to the physically most important case which is connected to spin–$\frac{1}{2}$–fields). All matrices are complex 4×4-matrices and $\widetilde{\alpha}^i$ and $\widetilde{\beta}$ obey $(\widetilde{\alpha}^i)^+ = \widetilde{\alpha}^i$, $\widetilde{\beta}^+ = \widetilde{\beta}$, and $\Gamma^+ = \Gamma + i\partial_i\widetilde{\alpha}^i$. However, they are not assumed to fulfill a Clifford algebra.

Here we introduced the coupling to the Maxwell field in the usual manner namely through minimal coupling $\nabla_i = \partial_i - \frac{ie}{c}A_i$. We also take the usual form of the Maxwell field as granted (for the experimental status of the electromagnetic field to obey EEP compare [10]). It is of course also possible to couple the Maxwell field in an anomalous way to the GDE, analogous to the $TH\epsilon\mu$–formalism. However, since any modification of this kind will result in corrections of the same structure as those which we will derive below, we will not take anomalous couplings to the Maxwell field into account.

We also introduced a c which has the dimension of a velocity. This velocity can be introduced by considering the null cones which the GDE defines: c is the maximum speed of propagation (from the physical point of view it is approximately the velocity of light, because any deviation from SRT, if there is any, is small). The purpose of this velocity is twofold: First, it makes the coefficient in front of the spatial derivative dimensionless (what is necessary in order to connect $\widetilde{\alpha}^i$ with space–time geometry), and second, it will be used later as ordering parameter in a Foldy–Wouthuysen transformation leading to the non–relativistic limit of the GDE.

The splitting between the matrices $c\Gamma$ and $mc^2\widetilde{\beta}$ is defined by means of a WKB approximation (compare [11]). While $mc^2\widetilde{\beta}$ is the “mass”–tensor which appears in the lowest order of approximation, Γ influences the first order only. Both matrices have the dimension of length^{-1}. In order to extract from the “mass”–tensor a dimensionless matrix possessing a geometrical meaning, we introduced a parameter m (so that mc^2 has the dimension time^{-1}) which can also be defined via the WKB approximation.

Eqn. (1) is general enough to describe violations of basic principles of GR. However, since (1) is a symmetric hyperbolic system very general principles of quantum mechanics are still fulfilled, namely (i) the well–posedness of the Cauchy problem, (ii) the superposition principle,

(iii) finite propagation speed, and (iv) a conservation law. Indeed, it has been shown that this GDE can be derived from these fundamental principles, see [8] for a review.

If we introduce the quantities

$$\begin{aligned}
\frac{4}{g^{00}} &:= \mathrm{tr}\widetilde{\beta}^2 &(2)\\
\mathbf{g}^{0i} &:= \frac{g^{0i}}{g^{00}} := \frac{1}{4}\mathrm{tr}(\widetilde{\alpha}^i) &(3)\\
\mathbf{g}^{ij} &:= -\frac{g^{ij}}{g^{00}} := \frac{1}{4}\mathrm{tr}(\widetilde{\alpha}^i\widetilde{\alpha}^j) - 2\mathbf{g}^{0i}\mathbf{g}^{0j} &(4)
\end{aligned}$$

then the matrices $\widetilde{\alpha}^i$ and $\widetilde{\beta}$ fulfill

$$\begin{aligned}
\widetilde{\alpha}^{(i}\widetilde{\alpha}^{j)} - \mathbf{g}^{ij}1 - 2\mathbf{g}^{0(i}\widetilde{\alpha}^{j)} &= X^{ij} &(5)\\
\widetilde{\alpha}^i\widetilde{\beta} + \widetilde{\beta}\widetilde{\alpha}^i - 2\mathbf{g}^{0i}\widetilde{\beta} &= 2X^i &(6)\\
\widetilde{\beta}^2 - \frac{1}{g^{00}} &= X &(7)
\end{aligned}$$

where the deviation from the usual Clifford algebra is described by the matrices X, X^i, and X^{ij}. (In the case $X = 0$, $X^i = 0$, and $X^{ij} = 0$ one can represent $\alpha^i = (\gamma^0)^{-1}\gamma^i$ and $\beta = (\gamma^0)^{-1}$ with matrices γ^μ fulfilling $\gamma^{(\mu}\gamma^{\nu)} = g^{\mu\nu}$, $\mu, \nu = 0, \ldots 3$. Even in the case that the X–matrices do not vanish it can be shown [11] that the matrices $\widetilde{\alpha}^i$ and $\widetilde{\beta}$ fulfill a generalised Clifford algebra.) If the matrices $\widetilde{\alpha}^i$ and $\widetilde{\beta}$ do not fulfill the usual Clifford algebra then the characteristic surfaces (null cones) and the mass shells (see Figure) of the generalised Dirac equation split and do not longer coincide with the usual light cones and mass shells. It is obvious that in these cases LLI is violated. This has also been discussed in [12] (see also [9] and [13] and references therein).

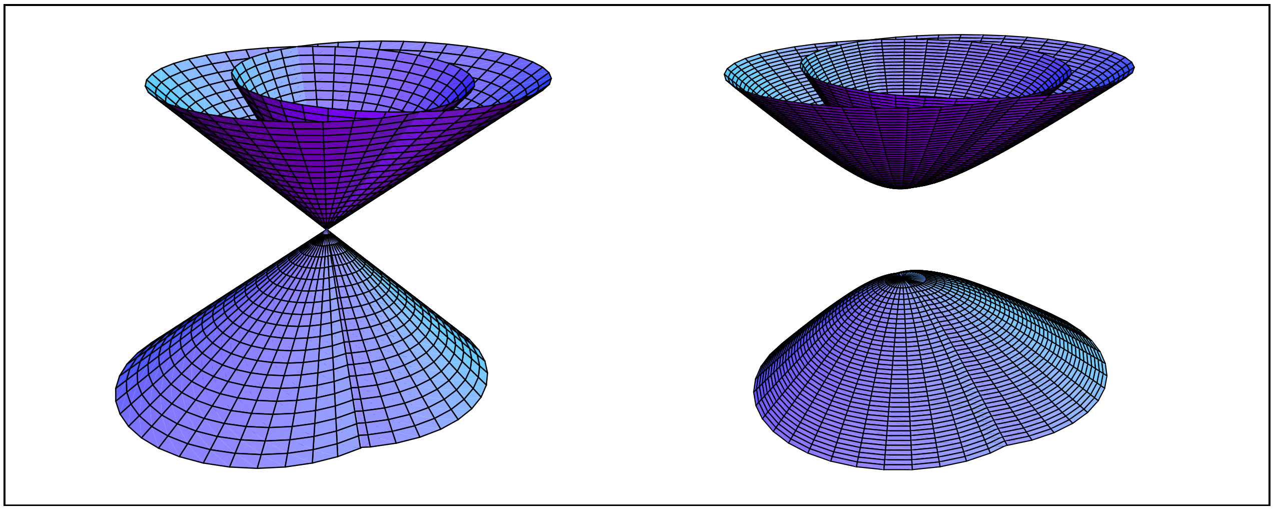
A velocity c can also be introduced by means of the quadratic form based on the tensor $g^{\mu\nu}$ defined by (2–4), which can be chosen to possess the signatur 2 compatible with the required propagation properties of the field φ.

2.2 Non–relativistic approximation: The generalised Pauli equation

While the Dirac equation is an equation for four spinorial components, a non-relativistic limit, the Pauli equation, has two components only. We have to eliminate two components which are small in the physical situation under consideration. The matrix which serves as tool for distiguishing the upper and lower (that is, the large and small) components is, as usual, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We also introduce as usual even and odd operators $\mathcal{E}$ and $\mathcal{O}$ which fulfill $\beta\mathcal{E} = \mathcal{E}\beta$ and $\beta\mathcal{O} = -\mathcal{O}\beta$.

First we split the matrices $\widetilde{\alpha}^i$ and Γ into an even and odd part: $\widetilde{\alpha}^i = \widetilde{\alpha}^i_e + \widetilde{\alpha}^i_o$ and $\Gamma = \Gamma_e + \Gamma_o$. We have $\dim(\widetilde{\alpha}^i) = \dim(\widetilde{\beta}) = 1$ and $\dim(\Gamma) = \dim(\partial_i)$. Then

$$\begin{aligned}
i\partial_t\varphi &= -ic(\widetilde{\alpha}^i_e + \widetilde{\alpha}^i_o)\nabla_i\varphi + c\Gamma\varphi + (\phi + mc^2\widetilde{\beta})\varphi \\
&= \beta mc^2 + \mathcal{E} + \mathcal{O} &(8)
\end{aligned}$$



with

$$\mathcal{O} = -ic\widetilde{\alpha}^i_o\nabla_i + c\Gamma_o + mc^2\widetilde{\beta}_o \tag{9}$$
$$\mathcal{E} = -ic\widetilde{\alpha}^i_e\nabla_i + c\Gamma_e + mc^2\widetilde{\beta}_e - mc^2\beta \tag{10}$$

Performing a Foldy-Wouthuysen-transformation (see, for example, [14]) with $\varphi' = U\varphi$, $U = e^{iS}$, $S = -\frac{i}{2m}\beta\mathcal{O}$ with $S^+ = S$ (since H as well as βmc^2 is hermitean we also know that $\mathcal{E}$ and $\mathcal{O}$ are hermitean) we get as resulting Hamiltonian

$$\begin{aligned} H'\varphi' &= \beta\left(mc^2 + \frac{\mathcal{O}^2}{2mc^2} - \frac{\mathcal{O}^4}{8m^3c^6}\right)\varphi' + \mathcal{E}\varphi' - \frac{1}{8m^2c^4}\left[\mathcal{O},[\mathcal{O},\mathcal{E}]\right]\varphi' - \frac{i}{8m^2c^4}[\mathcal{O},\dot{\mathcal{O}}]\varphi' \\ &= \beta mc^2\varphi' - \beta\frac{1}{2m}\widetilde{\alpha}^{(i}_o\widetilde{\alpha}^{i)}_o\nabla_i\nabla_j\varphi' \\ &\quad -\left[\beta\frac{1}{2m}(\widetilde{\alpha}^i_o\nabla_i\widetilde{\alpha}^j_o + i\{\Gamma_o,\widetilde{\alpha}^j_o\} + imc\{\widetilde{\beta}_o,\widetilde{\alpha}^j_o\}) + ic\widetilde{\alpha}^j_e\right]\nabla_j\varphi' \\ &\quad +\Big[\beta\Big(\frac{1}{2m}(\Gamma_o)^2 + \frac{1}{2}mc^2(\widetilde{\beta}_o)^2 + \frac{c}{2}\{\Gamma_o,\widetilde{\beta}_o\} - \frac{i}{2m}\widetilde{\alpha}^i_o\nabla_i(\Gamma_o + mc\widetilde{\beta}_o)\Big) \\ &\qquad +c\bar{M}^1_e + mc^2(\widetilde{\beta}_e - \beta) + \phi + \frac{i}{2m}\widetilde{\alpha}^{[i}_o\widetilde{\alpha}^{j]}_o H_{ij}\Big]\varphi' + \text{relativistic terms} \end{aligned} \tag{11}$$

with $H_{ij} = \partial_i A_j - \partial_j A_i$ and where we used a quasi–Newtonian coordinate system. With 'relativistic terms' we mean such terms which contain a c in the denominator. This Hamilton operator is of course hermitian since the operators $\mathcal{E}$ and $\mathcal{O}$ are hermitian.

We now make some specifications of the coefficients appearing in (11) in order to arrive at a physically interpretable generalised Pauli equation.

1. To start with, we make an ansatz concerning the introduction of the gravitational field. First, the gravitational field may consist of all coefficients appearing in (11). However, in the non–relativistic regime the main part of the gravitational field is given by the Newtonian potential $U(x)$ depending on the mass density $\rho(x)$ of the surrounding matter

distribution. In order to allow an anomalous coupling to the gravitational field the origin of which are mass distributions, we also take gravitational field tensors $U^{ij}(x)$ into account (see e.g [1]). There may also exist gravitational fields the origin of which are not mass distributions, but for example spin–sources which is the case in Einstein–Cartan theories, see [15]. Since these gravitational fields are expected to be very small we do not specify any spatial dependence of these fields: all gravitational fields which have their origin in other sources than in mass distributions are treated as constant. Therefore we assume that the x–dependence of the coefficients appearing in the above non–relativistic limit is only mediated by the Newtonian potential and the corresponding gravitational potential tensor [16, 1]. The remaining constant parts may be responsible for violations of LLI. Explicitly

$$\widetilde{\beta}(x) = \overset{0}{\widetilde{\beta}} + \overset{1}{\widetilde{\beta}}\frac{1}{c^2}\left(U(x) + \frac{\delta m'_{\mathrm{P}kl}}{m}U^{kl}(x)\right) \tag{12}$$

$$\widetilde{\alpha}^i(x) = \overset{0}{\widetilde{\alpha}}{}^i + \overset{1}{\widetilde{\alpha}}{}^i\frac{1}{c^2}\left(U(x) + \frac{\delta m'_{\mathrm{P}kl}}{m}U^{kl}(x)\right) \tag{13}$$

$$\Gamma(x) = \overset{0}{\Gamma} + \overset{1}{\Gamma}\frac{1}{c^2}\left(U(x) + \frac{\delta m'_{\mathrm{P}kl}}{m}U^{kl}(x)\right) + \overset{1}{\Gamma}{}^i\frac{1}{c^2}\left(\partial_i U(x) + \frac{\delta m'_{\mathrm{P}kl}}{m}\partial_i U^{kl}(x)\right) \tag{14}$$

where all coefficients $\overset{0}{\widetilde{\beta}}$, $\overset{1}{\widetilde{\beta}}$, $\overset{0}{\widetilde{\alpha}}{}^i$, $\overset{1}{\widetilde{\alpha}}{}^i$, $\overset{0}{\Gamma}$, $\overset{1}{\Gamma}$, and $\overset{1}{\Gamma}{}^i$ are constant matrices. While $\overset{0}{\widetilde{\beta}}$ and $\overset{0}{\widetilde{\alpha}}{}^i$ may be responsible for a violation of LLI, that is, for a violation of Special Relativity in vacuum, the other matrices may lead to U–induced violations of LLI and LPI.

In other words: after the specification of the dependence of the matrices $\widetilde{\beta}$, $\widetilde{\alpha}^i$, and Γ from the potential U and U^{ij} the remaining matrices $\overset{0}{\widetilde{\beta}}$, $\overset{0}{\widetilde{\alpha}}{}^{\hat{\mu}}$, and $\overset{0}{\Gamma}$ may also depend on x in a way not mediated by U. However, since we are going to describe experiments which are performed on a small scale only (atom beam interferometry, Huges–Drever experiments) any x–depencence which is not due to the Newtonian potential can be neglected and be replaced by the actual value of these matrices at the position of the experiment. Therefore, the above dependence on x is the only position–dependence of the corresponding matrices, that is $\widetilde{\beta}(x) = \widetilde{\beta}(U(x))$, and similarly for the matrices $\widetilde{\alpha}^{\hat{\mu}}(x)$ and $\Gamma(x)$.

Although we know from experience that U should be the Newtonian potential, at this stage of reasoning we do not need this specification so that at this place we take U and U^{ij} to be some unknown scalar and tensorial gravitational potentials. It is only at the last step where one makes a comparison with experiments that U is identified with the Newtonian potential and U^{ij} with the gravitational potential tensor.

The dependence on U^{ij} can be described by any parameter; we have chosen just for convenience and for staying in contact with the usual notation the combination $\delta m'_{\mathrm{P}ij}/m$.

The only remaining x-dependent terms are the gravitational potentials U, U^{ij}, the electromagnetic potentials ϕ, A_i, and the term H_{ij}.

2. Each anomalous term which does not vanish for $U = 0$ describes a possible LLI-violation. Consequently, all terms which remain after setting $U = 0$, are small. Therefore we can neglect the square $(\overset{0}{\Gamma}_o)^2$. The U-dependent terms indicate possible LPI-violations.

3. We make connection to the case of vanishing gravitational potential and non-rotating frame of reference: We take

$$\overset{0}{\mathbf{g}}{}^{ij} = \delta^{ij}, \qquad \overset{0}{\mathbf{g}}{}^{0i} = 0, \qquad \overset{0}{g}{}^{00} = 1 \tag{15}$$

and assume that these relations hold for all particles (neutrons, electrons, protons, etc).

4. We can further simplify the form of $\widetilde{\beta}$ of (12) by choosing a representation in which $\overset{0}{\widetilde{\beta}}$ is even: $\overset{0}{\widetilde{\beta}}_o = 0$.

5. We restrict to the 'large' component using the projection operator $P = \frac{1}{2}(1 + \beta)$. By doing so we introduce the following terms

$$P\overset{0}{\widetilde{\alpha}}{}_o^{(i}\overset{0}{\widetilde{\alpha}}{}_o^{j)} \quad =: \quad \delta^{ij} + \frac{\delta m_{\mathrm{I}}^{ij}}{m} + \frac{\delta \bar{m}_{\mathrm{I}k}^{ij}}{m}\sigma^k \tag{16}$$

$$P\overset{0}{\widetilde{\alpha}}{}_e^i \quad =: \quad A^i + A_j^i\sigma^j \tag{17}$$

$$P\{\overset{0}{\Gamma}_o, \overset{0}{\widetilde{\alpha}}{}_o^i\} \quad =: \quad 2(a^i + a_j^i\sigma^j) \tag{18}$$

$$P\overset{0}{\Gamma}_e \quad =: \quad T + T_j\sigma^j \tag{19}$$

$$P\overset{1}{\widetilde{\beta}}_e \quad =: \quad 1 + d + C_j\sigma^j \tag{20}$$

$$P(\overset{0}{\widetilde{\beta}}_e - \beta) \quad =: \quad B + B_j\sigma^j \tag{21}$$

$$P\overset{0}{\widetilde{\alpha}}{}_o^{[i}\overset{0}{\widetilde{\alpha}}{}_o^{j]} \quad =: \quad K^{ij} + ({\epsilon^{ij}}_k + K_k^{ij})\sigma^k \tag{22}$$

where σ^i are the usual Pauli matrices. The anomalous inertial mass tensor in (16) consists in two parts: a scalar part and a spin part. The hermiticity of $\widetilde{\alpha}^i$ ensures that $\delta m_{\mathrm{I}}^{ij}$ as well as $\delta \bar{m}_{\mathrm{I}k}^{ij}$ are real tensors. Note that $\delta m_{\mathrm{I}}^{ij}$ and $\delta \bar{m}_{\mathrm{I}k}^{ij}$ depends on $\widetilde{\alpha}^{\hat{\mu}}$ only, $\widetilde{\beta}$ has no influence on the kinetic term. We can absorb d into the anomalous gravitational mass tensor giving $\delta m_{\mathrm{P}ij}$.

For the usual Clifford algebra we have vanishing parameters $\delta m_{\mathrm{I}}^{ij}$, $\delta \bar{m}_{\mathrm{I}k}^{ij}$, A^i, A_j^i, d, C_j, B, B_j, K^{ij}, and K_k^{ij}. If any one of these parameters does not vanish, LLI is violated. The anomalous parameters which are connected with $\overset{0}{\Gamma}$ indicate a violation of LPI.

If space–time is endowed with a hypothetical torsion then the usual Dirac equation minimally coupled to metric and torsion gives rise to the quantities a_j^i and T_j. The latter is the space part of the axial torsion vector, and the first is related to the corresponding time component.

6. We transform away the term $\left(A^i + \frac{1}{m}a^i\right)\nabla_i\varphi'$ by means of $\varphi' = e^{-imc\delta_{ij}(A^i+\frac{1}{m}a^i)x^j}\widehat{\varphi}$. The resulting constant scalar parts appearing in the last term of (11) can be removed by an appropriate transformation.

We arrive at

$$\begin{aligned} H'\widehat{\varphi}' \;=\; & -\frac{1}{2m}\left(\delta^{ij} + \frac{\delta m_{\mathrm{I}}^{ij}}{m} + \frac{\delta\bar{m}_{\mathrm{I}k}^{ij}}{m}\sigma^k\right)\nabla_i\nabla_j\widehat{\varphi}' - \left(\frac{1}{m}a^i_j + cA^i_j\right)\sigma^j i\nabla_i\widehat{\varphi}' \\ & +\Big[e\phi(x) + \frac{e}{2m}H_i(x)(K^i + ({\epsilon^{ij}}_k + K^i_k)\sigma^k) \\ & +(mc^2B_i + cT_i)\sigma^i + (1 + C_i\sigma^i)mU(x) + \delta m_{\mathrm{P}ij}U^{ij}(x)\Big]\widehat{\varphi}' \end{aligned} \tag{23}$$

where $\phi(x)$ is the electrostatic potential and $H_i = \frac{1}{2}\epsilon_{ijk}H_{jk}$ the magnetic field. This is the GPE we looked for. All terms but the U, ϕ, H_i and the A_i which is part of the covariant derivative, are constant. The tensors $\delta m_{\mathrm{I}}^{ij}$ and $\delta\bar{m}_{\mathrm{I}k}^{ij}$ give spin dependent anomalous inertial mass tensors, a^i_j and A^i_j amount to a spin–momentum coupling, mc^2B_i may be considered as a spin–dependent "rest mass", the T_i may be interpreted as or identified with the space–like part of an axial torsion vector, and $\delta m_{\mathrm{P}ij}$ and C_i are anomalous spin dependent passive gravitational mass tensors. K^i and K^i_k give anomalous modifications of the coupling of the magnetic field to the spin–$\frac{1}{2}$ particle. Due to our systematic approach (23) contains all possible anomalous interactions on the non–relativistic level. The GPE (23) is a non–trivial generalisation of Haugan's [16] test theory for matter with spin.

A distinguished feature of (23) is the anomalous coupling of spin to the Newtonian potential. Such couplings have been considered first by Hari Dass [19]. However, our coupling of the spin to the Newtonian potential is different from the coupling Hari Dass [19] introduced for spherically symmetric Newtonian gravitational fields. While his couplings are of relativistic order on dimensional grounds, we obtained such a coupling by means of an additional anomalous property C_i of the quantum field. Due to this new structure it is possible to have a coupling of the spin to the Newtonian potential at non–relativistic order.

Note that there is no need and no possibility to introduce any $\hbar$. Indeed, also in the usual Schrödinger theory only the ratio $\hbar/m$ enters the equation of motion (see [17]). In this sense our mass–like parameters are all to be understood in the sense of being the ratio of mass and $\hbar$. Our mass-like parameter has the dimension of time/length^2, and our Hamilton operator has the dimension 1/time. It is no problem to introduce artificially an $\hbar$ so that the equations acquire the usual form and all parameters have the usual dimensions. Note also that B_i is dimensionless. T_i and a^i_j have the dimension 1/length. In the following we will neglect the coupling of the magnetic field to anomalous terms.

It should be emphasized that it is not possible to absorb the parameters $\left(\frac{1}{m}a^i_j + cA^i_j\right)\sigma^j$, B_i, and T_i into the inertial or gravitational spin–dependent anomalous mass tensors. Indeed, if we perform a transformation $\varphi \to \varphi' = S\varphi$, insert it into (23), and demand that the resulting coefficient of the first derivative should vanish, then we get a transformation matrix S which is linear in x. Such an x–dependence is not in accordance with a spherically symmetric Newtonian potential, for example. In addition, assuming a time–dependent transformation

necessarily leads to time–dependent coefficients, which makes a stationary problem non–stationary thus changing the structure of (23) dramatically.

Since (23) can be inferred from the very general GDE (1) all anomalous terms in (23) are derived in systematic manner. These are the most general anomalous terms on the non–relativistic level which can be derived from a GDE which is the most general equation obeying the very general basic principles listed above. The anomalous terms are necessarily connected with that parts of the matrices $\widetilde{\alpha}^i$, $\widetilde{\beta}$, and $\overset{0}{\Gamma}$ which are responsible for a possible violation of LLI and LPI. A violation of LLI is possible even if the gravitational potentials are turned off.

To sum up: we base our *quantum test theory* on a GPE (23). m is the usual scalar mass, ϕ and A_i the scalar and vector electromagnetic potential and H_i the magnetic part of the electromagnetic field. Thereby we assume that the electromagnetic interaction has the usual form, that is, there are no EEP–violating effects due to electromagnetism. $\delta m_{\mathrm{I}}^{ij}$ and $\delta\bar{m}_{\mathrm{I}k}^{ij}$ are anomalous inertial mass tensors where the latter is connected with the spin of the quantum system, and $\delta m_{\mathrm{P}ij}$ is the passive gravitational anomalous mass tensor. We introduced in addition a gravitational potential tensor U^{ij} with $\delta_{ij}U^{ij} = U$. A^i_j, B_i, and C_i are dimensionless constants. $\delta m_{\mathrm{I}}^{ij}$, $\delta\bar{m}_{\mathrm{I}k}^{ij}$, a^i_j, A^i_j, and B_i give rise to LLI–violation, while C_i and $\delta m_{\mathrm{P}ij}$ are responsible for LPI–violation. If all these coefficients vanish, we recover the usual Schrödinger equation coupled to the Newtonian potential. All coefficients are real. It is clear that with the energy mc^2 and the characteristic dimensionless quantity U/c^2 describing a gravitational interaction, the GPE is the most general 2nd order differential equation including spin and the gravitational potential tensor. Only U, U^{ij}, ϕ_i, A_i and H_{ij} are x–dependent. H is hermitian.

2.3 The classical limit

The corresponding classical Hamilton function is

$$\begin{aligned} H' &= \frac{1}{2m}\left(\delta^{ij} + \frac{\delta m_{\mathrm{I}}^{ij}}{m} + 2\frac{\delta\bar{m}_{\mathrm{I}k}^{ij}}{m}S^k\right)p_i p_j - 2\left(\frac{1}{m}a^i_j + cA^i_j\right)S^j p_i \\ &\quad + e\phi + \frac{e}{m}H_i S^i + 2(mc^2 B_i + cT_i)S^i + (1 + 2C_i S^i)mU + \delta m_{\mathrm{P}ij}U^{ij} \end{aligned} \tag{24}$$

where p_i is the momentum and S^i is the spin of the particle. The velocity, force and acceleration for vanishing electromagnetic field is

$$v^i = \frac{1}{m}\left(\delta^{ij} + \frac{\delta m_{\mathrm{I}}^{ij}}{m} + 2\frac{\delta\bar{m}_{\mathrm{I}k}^{ij}}{m}S^k\right)p_j - 2\left(\frac{1}{m}a^i_j + cA^i_j\right)S^j \tag{25}$$

$$f_i = -(1 + 2C_j S^j)m\partial_i U - \delta m_{\mathrm{P}kl}\partial_i U^{kl} \tag{26}$$

$$a^i = -\left(\delta^{ij} + \frac{\delta m_{\mathrm{I}}^{ij}}{m} + 2\left(\frac{\delta\bar{m}_{\mathrm{I}k}^{ij}}{m} + \delta^{ij}C_k\right)S^k\right)\partial_j U - \delta^{ij}\frac{\delta m_{\mathrm{P}kl}}{m}\partial_j U^{kl} \tag{27}$$

where we neglected the dynamics of thr spin vector. This is reasonable because the corresponding interaction terms are very small. It is obvious that this generalises Haugan's result [16] by introducing the spin S^k. In addition, a very important point which one can see by

comparing (27) with (23) is that in the GPE there are more LLI and LPI–violating parameters than in the acceleration (27) of the corresponding classical particle with spin, namely A^i_j and B_i. This acceleration is that quantity which is measured by Eötvös–type experiments, like the torsion balance experiments, or the proposed experiments of the Bremen–tower and STEP. (Therefore (27) is a frame to describe tests of the equivalence principle for macroscopic matter with polarisation.) That means: *If the classical acceleration fulfills LLI and LPI then this does not rule out the possibility of LLI and LPI–violating terms on the quantum level. On the quantum level there are more possibilities to violate EEP than on the classical level*, even if one includes polarised bulk matter.

However, by considering the dynamics of spin, one gets access to all the EEP violating parameters. For the dynamics of the spin expectation value in the classical approximation we get

$$\frac{d}{dt}\boldsymbol{S} = \boldsymbol{\Omega} \times \boldsymbol{S} \tag{28}$$

with

$$\Omega_i := \frac{1}{2m}\frac{\delta\bar{m}^{kl}_{\mathrm{I}i}}{m} p_k p_l + \left(\frac{1}{m}a^j_i + cA^j_i\right) p_i + mc^2 B_i + c\,T_i + C_i m U(x) \tag{29}$$

That means that besides $\delta m^{ij}_{\mathrm{I}}$ and $\delta m_{\mathrm{P}kl}$ all the anomalous parameters influence the spin precession. In other words: Only if one takes the path (27) *and* the dynamics of the spin (28) into account one can make statements about the complete set of parameters characterising the violation of EEP. However, the precession of the net polarisation of a macroscopic body is very difficult to observe. The corresponding quantum tests (see below) are much more sensitive.

It is important that *any deviation from the usual Schrödinger equation coupled to the Newtonian potential* will give rise to LLI– or LPI–violations. Consequently, *a non-vanishing of one of the above parameters implies that gravity is not describable by a Riemannian space-time metric.*

2.4 Possible consequences of LLI violating parameters

First we note that the coefficients T, T_i, and a^i_j are due to the term Γ which is not connected with the matrices $\widetilde{\alpha}^{\hat{\mu}}$ and $\widetilde{\beta}$ and therefore have no influence on the violation of LLI.

Since we performed a non-relativistic limit we are not able to reduce the generalised Clifford algebra by means of possible experiments indicating that the coefficients $\delta m^{ij}_{\mathrm{I}}$, $\delta\bar{m}^{ij}_{\mathrm{I}k}$, A^i_j, B_i, and C_i are zero. The only reasoning which can be done is that *if experiments indicate that one of the coefficients $\delta m^{ij}_{\mathrm{I}}$, $\delta\bar{m}^{ij}_{\mathrm{I}k}$, A^i_j, B_i, and C_i is non–vanishing then at least one of the X^{ij}, X^i, and X is non–vanishing also.*[1] In other words: any deviation from the usual Schrödinger equation coupled to the Newtonian potential will give rise to a splitting of the null cone or of the mass shell, and in turn, to a LLI– or LPI–violation.

That means, a non-vanishing of the above parameters necessarily implies that there is no usual Clifford algebra. Therefore, there is no Riemannian metric which can be defined

[1] Indeed, it is possible to show that (i) $B_i \neq 0 \Rightarrow X \neq 0$, (ii) $A^i_j \neq 0 \Rightarrow X^i \neq 0$, and irreducible tensor part of $\delta m^{ij}_{\mathrm{I}} \neq 0 \Rightarrow X^{ij} \neq 0$ (the trace results in a redefinition of the scalar mass m).

from the dynamics of the field under consideration and which is responsible for the dynamical behaviour of the quantum field. *A non-vanishing of one of the above parameters implies that gravity is not described by a Riemannian space—time metric.*

3 Experimental restrictions

3.1 Matter wave interferometry

We propose two different kinds of interference experiments where we use an interferometer of Kaservich and Chu type [20, 21]. The first is an experiment where only a spin-flip will be performed whereby both parts of the matter wave propagate with the same momentum. The second is an interference experiment which measures the acceleration.

3.1.1 Spin-flip experiment

We take an atomic beam with a definite spin value along a certain axis propagating with momentum p_i. We split this atomic state into two states and perform with one of these states two spin-flips, one at time t and the second one reverses the first flip at time $t + \Delta t$. The phase shift after the second splin-flip (for convenience, we introduce $\hbar$ in an obvious way) $\phi = \frac{1}{\hbar}(H(p, S) - H(p, -S))\Delta t$ is given by (for $A_i = \phi = 0$)

$$\phi = \frac{2}{\hbar}\left(\frac{\delta\bar{m}^{ij}_{\mathrm{I}k}}{2m^2}p_i p_j - \frac{\hbar}{m}a^i_k p_i - cA^i_k p_i + mc^2 B_k + C_k mU + cT_k\right) S^k \Delta t \tag{30}$$

To first order we can replace the momentum by the velocity, and we use $U = GM/r$. Then

$$\phi = \frac{1}{\hbar}\left(\frac{\delta\bar{m}^{ij}_{\mathrm{I}k}}{m}p_i\delta_{jk}l^k - 2(cA^i_j m + a^i_j)\delta_{ik}l^k + 2mc^2 B_j\Delta t + 2C_j m\frac{GM}{R}\Delta t + \hbar c T_j \Delta t\right) S^j \tag{31}$$

For an atom interferometer of Kasevich and Chu type we have $l \approx 1$ cm, $m \approx 10^{-26}$ kg, $v \approx 10$ cm/sec, $S = \frac{1}{2}$, and $\Delta t \approx 0.1$ sec. With the accuracy $\Delta\phi/\phi = 10^{-8}$ we can estimate in the case that one performs a null-experiment: $\left|\delta\bar{m}^{ij}_{\mathrm{I}k}/m\right| \lesssim 10^{-7}$, $\left|A^i_j\right| \lesssim 10^{-17}$, $|a^i_j| \lesssim 1$ m^{-1}, $|B_i| \lesssim 5\cdot 10^{-27}$, $|C_i| \lesssim 10^{-17}$, and $|T_i| \lesssim 3\cdot 10^{-10}$ m^{-1}. For the first coefficient we may get a better estimate if we take a large velocity $v = 10^3$ m/sec and $l = 100$ m. We get $\left|\delta\bar{m}^{ij}_{\mathrm{I}k}/m\right| \lesssim 10^{-15}$. This generalises results in [13] (see also [22]). If one of these quantities turns out to be non-null, then we infer a violation of LLI and LPI. However, all these quantities but the A^i_j and a^i_j can be measured better by Hughes–Drever type experiments (see below).

3.1.2 Measurement of acceleration

For the atom beam interferometer of Kasevich and Chu [20, 21] we get a phase shift $\phi = -k_i a^i T^2$ where T is the time between the laser pulses and k_i is the difference of the wave vectors of the two counterpropagating laser beams. We use the acceleration (27) assuming that the spin state is not affected by the beam splitting process. For a spherical symmetric

mass producing the gravitational field with constant mass density we get the phase shift (r_0^i is the vector from the center of the gravitating body (say, the earth) to the beam splitter)

$$\begin{aligned} \phi &= T^2 \frac{GM}{r_0^3} \left(k_i r_0^i + \frac{\delta m_{\mathrm{P}ij}}{m} \frac{r_0^i r_0^j}{r_0^2} k_l r_0^l - \frac{6}{5} \frac{\delta m_{\mathrm{P}ij}}{m} r_0^i k^j \right. \\ &\quad \left. + \frac{2}{5} \frac{\delta m_{\mathrm{P}ii}}{m} k_l r_0^l - \frac{\delta m_{\mathrm{I}ij}(S)}{m} r_0^i k^j \right) \end{aligned} \tag{32}$$

where $\delta m_{\mathrm{I}ij}(S)$ is the sum of both anomalous inertial mass tensors which depend on the spin. If we align $\boldsymbol{x}_0 \sim \vec{e}_z$ and denote the angle between r_0^i and k^i by ϑ then we get to first order in $\delta m_{\mathrm{P}ij}$

$$\delta\phi = -\left(1+\alpha\right) k\, T^2 g \cos\left(\vartheta + b\right) \tag{33}$$

with

$$\alpha := \frac{1}{5}\left(\frac{\delta m_{\mathrm{P}zz}}{m} + 4\frac{\delta m_{\mathrm{P}xx}}{m}\right) - \frac{\delta m_{\mathrm{I}zz}(S)}{m} \tag{34}$$

$$b := \frac{6}{5}\frac{\delta m_{\mathrm{P}zx}}{m} + \frac{\delta m_{\mathrm{I}zx}(S)}{m} \tag{35}$$

where $\delta m_{\mathrm{P}xx} = \delta m_{\mathrm{P}yy}$ due to the spherical symmetry of the gravitating body. α and b are constant for one sort of atoms. Note that (33) is an exact quantum result although there appears no $\hbar$. The reason is that only experimentally given quantities like k and T are used. The non-diagonal parts b induce a deviation from orthogonality: By choosing k_i orthogonal to $g^i \sim r_0^i$, the phase shift will not vanish in contrast to the case $b = 0$. Instead we have to first order

$$\phi = bkT^2 g\,. \tag{36}$$

The phase shift will vanish only for an angle $\vartheta = -(6/5)\delta m_{\mathrm{P}zx}/m - \delta m_{\mathrm{I}zx}(S)/m$. The diagonal parts α modify the amount of the phase shift for given k and flight time T and are best measured if $k^i \sim g^i$. The phase shift is to first order

$$\phi = -(1+\alpha)kT^2 g\,. \tag{37}$$

In order to measure α and b we consider two cases: (i) $\vartheta = 0$ with the phase shift (37) for measuring α and (ii) $\vartheta = \pi/2$ with the phase shift (36) for measuring b where we use $g := \nabla U = GM_\oplus/r_0^2$ as acceleration of the earth's gravitational field at the position of the beam splitter.

A possible way to measure α is to consider the variation of the phase shift (37) by varying the elevation of the whole interferometer with respect to the surface of the earth, that is, by varying r from $r_0 = R_\oplus$ to $r_1 = r_0 + \delta r$. This procedure is especially appropriate for the Raman light- pulse atom beam interferometer because it may be possible to detect by means of phase shifts the difference in the elevation of 1mm. This corresponds to an accuracy of the measured phase shift of $\Delta\phi/\phi \approx 10^{-10}$ which [21] claim to be possible to achieve. The today's accuracy is 10^{-8}. Using this experimental setup one gets estimates for α by detecting deviations of the measured phase shift from $\delta\phi_0 = kT^2\Delta g \cos\vartheta$ where $\Delta g = g(R_\oplus{+}\delta r){-}g(R_\oplus)$ is the difference of the acceleration between $R_\oplus + \delta r$ and $R_\oplus$. Thereby the acceleration or the

earth's gravitational potential is measured by gravimeters (or satellites) which are of other material composition than the interfering atoms. Therefore we actually compare the possible composition dependence of the gravitational acceleration between the atom beam and other gravimeters.

In principle the energy levels of the atoms are also modified due to a violation of LPI and LLI (see [24, 25] for calculations within the $TH\epsilon\mu$–formalism and below). However, this does not matter since the wave vector and frequency of the laser beam which are related to the difference of the energy levels are given quantities in the quasi Newtonian coordinate system. Consequently, we can use formula (33) with the actually used momentum k to determine α: $1+\alpha = \phi/kT^2\Delta g$ where ϕ is the measured variation of the phase shift during elevation of the interferometer from r_0 to $r_0+\delta r$. If one assumes a null experiment, then the accuracies of the various entities entering the phase shift will give an estimate of the value of α which depends on the accuracy of ϕ, kT^2, and Δg. The first two have been estimated in [20, 21] to $3\cdot 10^{-8}$ and the best absolute gravimeters have an accuracy of 10^{-6}. Consequently, an observation of the phase shift by elevating the whole interferometer about $\delta r = 100$m and a null result will give $|\alpha| \lesssim 3\cdot 10^{-6}$ and consequently $\left|\frac{\delta m_{\mathrm{P}xx}}{m} + 4\frac{\delta m_{\mathrm{P}zz}}{m} - 5\frac{\delta m_{\mathrm{I}zz}(S)}{m}\right| \lesssim 1.5\cdot 10^{-5}$. With the results of Hughes–Drever type experiments (see below) these experiments amount to an estimate $|\delta m_{\mathrm{P}ij}/m| \lesssim 10^{-7}$ which is usually tested by red–shift experiments to $\lesssim 10^{-4}$, see [1, 23], for example.

A second way to measure this effect is to take a second apparatus with a different kind of atoms, elevate the two apparatus, and to compare the phase shifts. Here one does not need a measurement of the gravitational acceleration by some gravimeter. In this case one measures the difference $\alpha_1 - \alpha_2$ with, for a null result, an accuracy given by the atom beam interferometers $|\alpha_1 - \alpha_2| \lesssim 6\cdot 10^{-8}$.

A third way to measure α with (37) is to stay with the apparatus at the same position and vary T which leads approximately to the same estimate for α and $\delta m_{\mathrm{P}}/m$ as above.

For measuring b we have to align k_i orthogonal to g^i. For a given initial velocity of the atoms and a specially chosen time T one can arrange that the atoms may fly parts of parabolae so that the $\pi/2$ pulses interact with the beams on the same height and the π pulse hits both beams at their peak. If the experiment gives a null result then the accuracy of b is limited by the accuracy of fixing k_i exactly orthogonal to g^i. Since this accuracy is approximately 10^{-6} (again given by satellite results) we have $|b| \lesssim 10^{-6}$ and consequently $|\delta m_{\mathrm{P}zx}/m + (5/6)\delta m_{\mathrm{I}zx}(S)/m| \lesssim 10^{-6}$.

3.2 Hughes-Drever experiment

We can also use the above Hamiltonian to calculate the Zeeman–splitting of energy levels in an atom. In order to do that we first have to consider a two-particle system and to eliminate the center–of–mass motion. In the case of a heavy nucleus, we can neglect the difference between the mass and the reduced mass. With the single particle Hamiltonian (23) and fixing the nucleus at $x=0$ we get the Hamiltonian describing the energy levels of an electromagnetically bound system:

$$H_{\mathrm{I}} = H_0 + H_{\mathrm{I,\,em}} + H_{\mathrm{I,\,non\text{-}Einst.}} \tag{38}$$

with

$$
\begin{aligned}
H_0 &= -\frac{\hbar^2}{2m}\Delta + \frac{Ze^2}{|x|} + V_{\rm nucl}(x) && (39)\\
H_{\rm I,\,em} &= \frac{e}{2m}H_i(L^i + \sigma^i) && (40)\\
H_{\rm I,\,non\text{-}Einst.} &= -\frac{\hbar^2}{2m}\frac{\delta m_{\rm I}^{ij}(\sigma)}{m}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + \left(\frac{1}{m}a^i_j + cA^i_j\right)\sigma^j i\hbar\frac{\partial}{\partial x^i} - \hbar c T_i\sigma^i + c^2 m B_i\sigma^i \\
&\quad + mC_i\sigma^i U - \delta m_{\rm I}^{ij}(\sigma)x_i\nabla_j U - \delta m_{Pkl}x^i\nabla_i U^{kl} && (41)
\end{aligned}
$$

and where $V_{\rm nucl}$ is the nuclear binding potential. Here x is the relative coordinate between e.g. the core of a nucleus and a valence proton. Note that there are no Einsteinian effects due to the acceleration ∇U. This is in accordance with the equivalence principle: The effect of gravitational acceleration can be cancelled by a transformation to a suitable accelerated frame. H_0 describes the atom without external fields and without disturbances, $H_{\rm I,\,em}$ is the interaction of the electron with an external constant magnetic field, and $H_{\rm I,\,non\text{-}Einst.}$ gives LLI and LPI-violating effects. Terms linear in x^i do not contribute in first order to energy shifts. (38) is the equation for the energy levels.

We can consider the case of an atomic nucleus which consists of a core and a valence proton which is considered in the usual Hughes-Drever experiments (see e.g. [1]). In the case of ^{7}Li we have a $J = 0$ core and a valence proton with angular momentum $L = 1$ and spin $\frac{1}{2}$. In [28] and [26] ^{201}Hg and ^{21}Ne with a similar nuclear structure was used. With the wave functions $|J, M_J\rangle$ for the magnetic quantum numbers $M_J = \left\{\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right\}$ the matrix elements of $H_{\rm I,\,non\text{-}Einst.}$ are easily calculated whereby the matrix elements linear in momentum and in the relative position vanish. Therefore, the energy levels for the hyperfine-splitting are shifted and the singlet line thus splits into three lines with the energies

$$
\begin{aligned}
E(\tfrac{3}{2},\tfrac{3}{2}) - E(\tfrac{3}{2},\tfrac{1}{2}) &= \delta + \bar\delta_1 + \bar\delta_2 && (42)\\
E(\tfrac{3}{2},\tfrac{1}{2}) - E(\tfrac{3}{2},-\tfrac{1}{2}) &= \bar\delta_2 && (43)\\
E(\tfrac{3}{2},-\tfrac{1}{2}) - E(\tfrac{3}{2},-\tfrac{3}{2}) &= -\delta + \bar\delta_1 + \bar\delta_2 && (44)
\end{aligned}
$$

with

$$
\begin{aligned}
\delta &:= -\frac{\hbar^2}{3a^2}\left(\frac{\delta m_{\rm I}^{xx}}{m^2} + \frac{\delta m_{\rm I}^{yy}}{m^2} - 2\frac{\delta m_{\rm I}^{zz}}{m^2}\right) && (45)\\
\bar\delta_1 &:= -\frac{\hbar^2}{a^2}\left(\frac{\delta \bar m_{{\rm I}z}^{xx}}{m^2} + \frac{\delta \bar m_{{\rm I}z}^{yy}}{m^2} - 2\frac{\delta \bar m_{{\rm I}z}^{zz}}{m^2}\right) && (46)\\
\bar\delta_2 &:= -\frac{5\hbar^2}{3a^2}\frac{\delta \bar m_{{\rm I}z}^{zz}}{m^2} + \frac{2}{3}(mc^2 B_z + mC_z U) + c\hbar T_z && (47)
\end{aligned}
$$

where we modelled $V_{\rm nucl}$ by the harmonic oscillator potential. a is the radius of the nucleus. δ and $\bar\delta_1$ is responsible for a splitting of the single line into three lines, and $\bar\delta_2$ shifts all three lines in the same way. The search for these splittings during the change of the z-axis with respect to the nonrotating Newtonian coordiante system amounts to a test of LLI and LPI–violation. Using the experimental accuracy of this type of experiment (see [26], also [27, 28]) we have

$\delta E < 0.45 \cdot 10^{-6}$ Hz so that, provided no unfortunate cancellation of terms occurs, we get the estimates $|\delta \bar{m}^{xx}_{\mathrm{I}z}/m| \lesssim 5 \cdot 10^{-30}$, $|B_z| \lesssim 3 \cdot 10^{-31}$, $|C_z| \lesssim 3 \cdot 10^{-24}$, and $|T_z| \lesssim 1.5 \cdot 10^{-15}\,\mathrm{m}^{-1}$ where we used $a = 1.5$ fm and the gravitational potential $U/c^2 \approx 10^{-7}$ of our galaxy. The estimate on T_i may be interpreted as the till now tightest constraint on a hypothetical axial torsion [29].

Since only the relative coordinate x appears in (41) our cause of violation of LLI is different from that described by the $TH\epsilon\mu$-formalism [6, 1]: While in the latter case the LLI-violation is due to a relative velocity with respect to the rest frame of the universe, it is in our case due to the anomalous mass tensors appearing in the GPE. These LLI-violating terms cannot be transformed away by choosing a special frame of reference.

4 Conclusions and Discussion

We have presented a general approach to a Pauli equation describing all possible anomalous couplings of the matter field to the Newtonian gravitational field and to hypothetical other gravitational fields. In deriving this equation we did not use any theoretical concept which has no direct physical interpretation nor any geometrical notion. The latter should be a consequence of experiments. Therefore we obtained in a systematic manner all anomalous couplings on the non-relativistic level, thus enlarging in a non–trival way Haugan's test theory. Main features of our quantum test theory are that we include spin in a non–trivial way and that it possesses more possibilities to violate LLI and LPI than the corresponding classical theory.

We want to remark that the $TH\epsilon\mu$–formalism leads to anomalous coupling terms which are part of our set of anomalous terms derived above. Consequently, if one of these parameters does not vanish, then it needs further experimental and theoretical analysis in order to distinguish whether this anomalous term is due to an underlying GDE considered in this work, or due to an anomalous coupling between the quantum field and the Maxwell field.

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References

[1] Will C.M.: *Theory and Experiment in Gravitational Physics*, Revised Edition, Cambridge University Press, Cambridge (1993).

[2] Will C.M.: Testing Machian Effects in Laboratory and Space, **in:** Barbour J., Pfister H. (eds.): *Mach's Principle, From Newton's Bucket to Quantum Gravity*, Birkhäuser, Boston 1995.

[3] Robertson H.P.: Postulates versus Observations in the Special Theory of Relativity, *Rev. Mod. Phys.* **21**, 378 (1949).

[4] Mansouri R., Sexl R.U. :A Test Theory of Special Relativity: I. Simultaneity and Clock Synchronisation, *Gen. Rel. Grav.* **8**, 497 (1977); II. First Order Tests, *Gen. Rel. Grav.* **8**, 515 (1977); III. Second Order Tests, *Gen. Rel. Grav.* **8**, 809 (1977).

[5] Tourrenc Ph., Melliti T., Bosredon J.: A Covariant Frame to Test Special Relativity Including Relativistic Gravitation, *Gen. Rel. Grav.* **28**, 1071 (1996).

[6] Lightman A.P., Lee D.L.: Restricted Proof that the Weak Equivalence Principle Implies the Einstein Equivalence Principle, *Phys. Rev.* **D 8**, 364 (1973).

[7] Bordé Ch.J.: Atomic interferometry with internal state labeling, *Phys. Lett.* **A 140**, 10 (1989).

[8] Audretsch J., Lämmerzahl C.: A New Constructive Axiomatic Scheme for the Geometry of Space-Time, **in:** Majer U., Schmidt H.-J. (Edts.): *Semantical Aspects of Space-Time Geometry*, BI Verlag, Mannheim 1993, p.21.

[9] Bleyer U., Liebscher D.–E.: Mach's Principle and Causal Structure, **in:** Barbour J., Pfister H. (Edts.): *Mach's Principle, From Newton's Bucket to Quantum Gravity*, Birkhäuser, Boston 1995, p.293.

[10] Haugan M.P., Kauffmann T.F.: A New Test of the Einstein Equivalence Principle and the Isotropy of Space, *Phys. Rev.* **D 52**, 3168 (1995).

[11] Lämmerzahl C.: The Geometry of Matter Fields, **in:** deSabbata V., Audretsch J. (eds.): *Quantum Mechanics in Curved Space–Time*, NATO ASI Series, Series B: Physics, Vol. 230, Plenum Press, New York 1990, p.23/

[12] Liebscher D.-E.: The Geometry of the Dirac equation, *Ann. Physik* (Leipzig) **42** 35, (1985).

[13] Audretsch J., Bleyer U., Lämmerzahl C.: Testing Lorentz invariance with atomic beam interferometry, *Phys. Rev.* **A 47**, 4632 (1993).

[14] Bjorken J.D., Drell S.D. (1964): *Relativistic Quantum Mechanics*, McGraw-Hill, New York.

[15] Hehl F.W., von der Heyde P., Kerlick G.D. (1976): General relativity with spin and torsion: Foundations and prospects, *Rev. Mod. Phys.* **48**, 393.

[16] Haugan M.P.: Energy Conservation and the Principle of Equivalence, *Ann. Phys.* **118**, 156 (1979).

[17] Weiss D.S., Young B.C., Chu S: Precision measurement of $\hbar/m_{\mathrm{Cs}}$ based on photon recoil using laser–cooled atoms and atomic interferometry, *Appl. Phys.* **B 59**, 217 (1994).

[18] Bleyer U., Liebscher D.-E. **in:** Ref. [2]; Audretsch J., Lämmerzahl C.: A New Constructive Axiomatic Approach to Space-Time Structure, **in** Majer U., Schmidt K.-H.: *Semantical Aspects of Space-Time Geometry*, BI Verlag, Mannheim (1992).

[19] Hari Dass N.D.: Experimental Tests for Some Quantum Effects in Gravitation, *Ann. Physics (N.Y.)* **107**, 337 (1977).

[20] Kasevich M., Chu S.: Atomic Interferometry using stimulated Raman Transitions, *Phys. Rev. Lett.* **67**, 181 (1991).

[21] Kasevich M., Chu S.: Measurement of the gravitational acceleration of an atom with a light-pulse atom interferometer, *Appl. Phys.* **B 54**, 321 (1992).

[22] Phillips P.R.: Test of Spatial Isotropy Using a Cyrogenic Torsion Pendulum, *Phys. Rev. Lett.* **59**, 1784 (1987).

[23] Godone A., Novero C., Tavella P.: Null gravitational redshift experiments with nonidentical atomic clocks, *Phys. Rev.* **D 51**, 319 (1995).

[24] Will C.M.: Gravitational red–shift measurements as tests of nonmetric theories of gravity, *Phys. Rev.* **D 10**, 2330 (1974).

[25] Gabriel M.D., Haugan M.P.: Testing the Einstein equivalence principle: Atomic clocks and local Lorentz invariance, *Phys. Rev.* **D 41**, 2943 (1990).

[26] Chupp T.E., Hoara R.J., Loveman R.A., Oteiza E.R., Richardson J.M., Wagshul M.E. (1989): Results of a New Test of Local Lorentz Invariance: A Search for Mass Anisotropy in ^{21}Ne, *Phys. Rev. Lett.* **63**, 1541.

[27] Prestage J.D., Bollinger J.J., Itano W.M., Wineland D.J. (1985): Limits for Spatial Anisotropy by Use of Nuclear–Spin–Polarized $^{9}Be^{+}$ Ions, *Phys. Rev. Lett.* **54**, 2387.

[28] Lamoreaux S.K., Jacobs J.P., Heckel B.R., Raab F.J., Fortson E.N. (1986): New Limits on Spatial Anisotropy from Optically Pumped ^{201}Hg and ^{199}Hg, *Phys. Rev. Lett.* **57**, 3125.

[29] Lämmerzahl C.: Constraints on Space–Time Torsion from Hughes–Drever Experiments, to appear in *Phys. Lett. A*.